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On pseudo-Euclidean Novikov algebras

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Overview

Left-symmetric algebras

Definition

A left-symmetric algebra A *is a vector space over a field* $\mathbb{K}$ *with a bilinear product* $(x, y) \longrightarrow xy$ *satisfying* $(x, y, z) = (y, x, z)$ *for* $x, y, z \in \mathrm{A}$*, where* $(x, y, z) = (xy)z - x(yz)$.

Remark

If A *is a left-symmetric algebra, then the commutator* $[x, y] = xy - yx$ *define a Lie bracket on* A*. We denote this underlying Lie algebra by* $\mathrm{A_L}$.

Left-symmetric algebras are non-associative algebras arising from the study of affine manifolds and some other geometric structures. For example, a Lie group admits an affine connexion if and only if its Lie algebra admits a Left-symmetric product compatible with the Lie brackets.

Left-symmetric algebras

Let A be a left-symmetric algebra. We denote by L_x (resp. R_x) the left (resp. right) multiplication by x; i-e, $\mathrm{L}_x(y) = xy = \mathrm{R}_y(x)$.

Definition

A left-symmetric algebra is called transitive (or complete) if all right multiplications R_x are nilpotent.

The transitivity corresponds to the completeness of the affine manifolds in geometry.

Novikov algebras

Novikov algebras constitute a special class of left-symmetric algebras.

Definition

A Novikov algebra A *is a left-symmetric algebra such that all right multiplications commute. That is* $[\mathrm{R}_x, \mathrm{R}_y] = 0$ *for any* $x, y \in \mathrm{A}$.

The abstract study of Novikov algebras was started by Zelmanov and Filipov. The term "Novikov algebra" was given by Osborn.
Novikov asked whether there exist simple Novikov algebras.

Theorem (Zelmanov, Soviet Math. Dokl. 1987)

A finite-dimensional simple Novikov algebra over an algebraically closed field with characteristic 0 is one-dimensional.

Theorem (Burde, J. Geom. Phys. 2006)

The underlying Lie algebra of a Novikov algebra is solvable.

Novikov algebras with an invariant symmetric bilinear form

Definition

A nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ *on a Novikov algebra* A *is said to be invariant if* $\langle \mathrm{R}_x(y), z\rangle = \langle y, \mathrm{R}_x(z)\rangle$ *for any* $x, y, z \in \mathrm{A}$.

Let $\mathrm{A}_{k,0}$ denote the real vector space of dimension k with zero multiplication and $F_{k+1,0} = e_0, e_1, ..., e_k : e_i e_0 = ei, i = 0, ..., k$.

Theorem (Zelmanov, Sov. Math. Dokl. 1987)

Let A *be a real Novikov algebra provided with an invariant positive definite symmetric bilinear form. Then* A *is an orthogonal direct sum of the form* $\mathrm{A} = \oplus_i \mathrm{A}_i$ *where each* A_i *is isomorphic to either the algebra* $\mathrm{A}_{k,0}$ *or the algebra* $F_{k+1,0}$*, for some integer* $k \geq 1$*. In particular,* A *is associative.*

Novikov algebras with an invariant symmetric bilinear form

Theorem (Guediri, J. Geom. Phys. 2014 and 2016)

Let A *be a real n-dimensional Novikov algebra provided with an invariant Lorentzian symmetric bilinear form. Then* A *is isomorphic to an orthogonal direct sum of the form* $\mathrm{A} = \mathrm{A}_1 \oplus \mathrm{A}_2$*, where* A_1 *is an algebra in Table 1 and* A_2 *is a direct sum of the algebras* $A_{k,0}$ *and* $F_{k+1,0}$*.*

Pseudo-Euclidean Novikov algebras

Definition

A pseudo-Euclidean Novikov algebra $(\mathrm{A}, \langle\ ,\ \rangle)$ *is a Novikov algebra* A *with a non-degenerate symmetric bilinear form* $\langle\ ,\ \rangle$ *such that* $\langle \mathrm{L}_x(y), z\rangle + \langle y, \mathrm{L}_x(z)\rangle = 0$ *for any* $x, y, z \in \mathrm{A}$.

Definition

A flat pseudo-Euclidean Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ *is a Lie algebra* $\mathfrak{g}$ *with a non-degenerate symmetric bilinear form* $\langle\ ,\ \rangle$ *such that the Levi-Civita product defined by*

$$2\langle xy, z\rangle = \langle [x, y], z\rangle - \langle [y, z], x\rangle + \langle [z, x], y\rangle$$

is left-symmetric.

Pseudo-Euclidean Novikov algebras

Remark

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Novikov algebra. Then, $(\mathrm{A_L}, \langle\ ,\ \rangle)$ is a flat pseudo-Euclidean Lie algebra. Conversely, If $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a flat pseudo-Euclidean Lie algebra such that $[\mathrm{R}_x, \mathrm{R}_y] = 0$ for any $x, y \in \mathfrak{g}$ (with respect to the Levi-Civita product) then the vector space $\mathfrak{g}$ endowed by the Levi-Civita product and the bilinear form $\langle\ ,\ \rangle$ is a pseudo-Euclidean Novikov algebra.

Problem

Which Novikov algebras admit such pseudo-Euclidean metric? Or, equivalently, Which flat pseudo-Euclidean Lie algebras admit Novikov Levi-Civita product?

Reference and main result

Hicham Lebzioui (2020)
On pseudo-Euclidean Novikov algebras
Journal of Algebra 564, 300 – 316.

Theorem

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ be a Lorentzian Novikov algebra. Then A is transitive and A_{L} is unimoduar.

Double extension of flat pseudo-Euclidean Lie algebras

Let us recall the double extension process introduced by A. Aubert and A. Medina, and which plays an important role in the study of flat pseudo-Riemannian Lie groups. Let $(B, [\ ,\ ]_0, \langle\ ,\ \rangle_0)$ be a flat pseudo-Euclidean Lie algebra, $\xi, D : B \longrightarrow B$ two endomorphisms of B, $b_0 \in B$ and $\mu \in \mathbb{R}$ such that:

1. ξ is a 1-cocycle of $(B, [\ ,\ ]_0)$ with respect to the representation $\mathrm{L} : B \longrightarrow \mathrm{End}(B)$ defined by the left multiplication associated to the Levi-Civita product, i.e., for any $a, b \in B$,
$$\xi([a, b]) = \mathrm{L}_a \xi(b) - \mathrm{L}_b \xi(a), \tag{1}$$
2. $D - \xi$ is skew-symmetric with respect to $\langle\ ,\ \rangle_0$,
$$[D, \xi] = \xi^2 - \mu\xi - \mathrm{R}_{b_0}, \tag{2}$$
and for any $a, b \in B$
$$a.\xi(b) - \xi(a.b) = D(a).b + a.D(b) - D(a.b). \tag{3}$$

We call (μ, D, ξ, b_0) satisfying the two conditions above *admissible*.

Double extension of flat pseudo-Euclidean Lie algebras

Given (μ, D, ξ, b_0) admissible, we endow the vector space $\mathfrak{g} = \mathbb{R}e \oplus B \oplus \mathbb{R}\bar{e}$ with the inner product $\langle \ , \ \rangle$ which extends $\langle \ , \ \rangle_0$, for which $\mathrm{span}\{e, \bar{e}\}$ and B are orthogonal, $\langle e, e\rangle = \langle \bar{e}, \bar{e}\rangle = 0$ and $\langle e, \bar{e}\rangle = 1$. We define also on $\mathfrak{g}$ the brackets

$$[\bar{e}, e] = \mu e, \ [\bar{e}, a] = D(a) - \langle b_0, a\rangle_0 e \quad \text{and} \quad [a, b] = [a, b]_0 + \langle(\xi - \xi^*)(a), b\rangle_0 e,$$

where $a, b \in B$. Then $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is a flat pseudo-Euclidean Lie algebra called *double extension* of $(B, [\ ,\ ]_0, \langle\ ,\ \rangle_0)$ according to (μ, D, ξ, b_0). Conversely, if a flat pseudo-Euclidean Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ contains a totally isotropic two-sided ideal I (for the Levi-Civita product) such that $\dim \mathrm{I} = 1$ and $\mathrm{I}^\perp$ is also a two sided-ideal, then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is obtained by this process.

Double extension of flat pseudo-Euclidean Lie algebras

Note that, if we denote the Levi-Civita product in $\mathfrak{g}$ (resp. B) by $a.b$ (resp. ab), then we have for any $a, b \in B$,

$$\begin{cases} e.e &= e.a = a.e = e.\bar{e} = 0 \\ a.b &= \langle \xi(a), b \rangle_B e + ab \\ \bar{e}.e &= \mu e \\ \bar{e}.\bar{e} &= b_0 - \mu \bar{e} \\ \bar{e}.a &= -\langle b_0, a \rangle_B e + (D - \xi)(a) \\ a.\bar{e} &= -\xi(a) \end{cases} \tag{4}$$

Examples of pseudo-Euclidean Novikov algebras

Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Lie algebra. If $\mathfrak{g}$ splits orthogonaly as $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}$, where $\mathfrak{b}$ is an abelian sub-algebra, $\mathfrak{u}$ is an abelian ideal and ad_b is skew-symmetric for any $b \in \mathfrak{b}$, then $\mathfrak{g}$ endowed with the Levi-Civita product is a pseudo-Euclidean Novikov algebra.

Proof.

Let $x \in \mathfrak{u}$. If $y, z \in \mathfrak{u}$, then $\langle \mathrm{L}_x(y), z\rangle = 0$. If $y \in \mathfrak{u}$ and $z \in \mathfrak{b}$, since ad_z is skew-symmetric then $\langle \mathrm{L}_x(y), z\rangle = 0$. Thus $\mathrm{L}_x(y) = 0$ for any $y \in \mathfrak{u}$. Similarly, we show that $\mathrm{L}_x(y) = 0$ for any $y \in \mathfrak{b}$. Thus $\mathrm{L}_x = 0$ and $\mathrm{R}_x = -\mathrm{ad}_x$ for any $x \in \mathfrak{u}$. We have in the same way, $\mathrm{L}_x = \mathrm{ad}_x$ and $\mathrm{R}_x = 0$ for any $x \in \mathfrak{b}$. Thus $\mathrm{L}_{[x,y]} = [\mathrm{L}_x, \mathrm{L}_y]$ for any $x, y \in \mathfrak{g}$, which implies that the Levi-Civita product is left-symmetric. On the other hand, if $x \in \mathfrak{b}$ and $y \in \mathfrak{g}$, then we have obviously $[\mathrm{R}_x, \mathrm{R}_y] = 0$. Since $\mathfrak{u}$ is abelian, then for any $x, y \in \mathfrak{u}$, $[\mathrm{R}_x, \mathrm{R}_y] = [\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_{[x,y]} = 0$, which shows that the Levi-Civita product is of Novikov. □

Examples

1. *If a Lie group admits a flat left-invariant Riemannian metric then its Lie algebra endowed with the Levi-Civita product is an Euclidean Novikov algebra (Milnor, Adv. in Maths. 1976)*
2. *If a Lie group admits a flat left-invariant pseudo-Riemannian metric such that* $\langle\ ,\ \rangle_{[\mathfrak{g},\mathfrak{g}]}$ *is positive or negative definite then* $\mathfrak{g}$ *endowed with the Levi-Civita product is a pseudo-Euclidean Novikov algebra (L, Proc. of A.M.S. 2020)*

Theorem

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ *be a pseudo-Euclidean Novikov algebra. Then* $\langle\ ,\ \rangle_{[\mathrm{A_L},\mathrm{A_L}]}$ *is positive or negative definite if and only if* $\mathrm{A_L}$ *splits orthogonaly as* $\mathrm{A_L} = \mathfrak{b} \oplus \mathfrak{u}$, *where* $\mathfrak{b}$ *is an abelian sub-algebra,* $\mathfrak{u}$ *is an abelian ideal and* ad_b *is skew-symmetric for any* $b \in \mathfrak{b}$. *In this case, the Novikov product is given by* $\mathrm{R}_b = 0$ *for any* $b \in \mathfrak{b}$ *and* $\mathrm{R}_u = -\mathrm{ad}_u$ *for any* $u \in \mathfrak{u}$.

The following example gives a Lorentzian Novikov algebra $(\mathrm{A}, \langle\ ,\ \rangle)$ such that $[\mathrm{A_L}, \mathrm{A_L}]$ is Lorentzian.

Example

The Lie algebra $\mathfrak{e}(1,1)$ of the Lie group of rigid motions of Minkowski plane. $\mathfrak{e}(1,1) = \mathrm{span}\{e_1, e_2, e_3\}$ endowed with the only non vanishing Lie brackets $[e_1, e_2] = e_3$ and $[e_1, e_3] = e_2$ and the flat Lorentzian metric $\langle\ ,\ \rangle$ where $\{e_1, e_2, e_3\}$ is orthonormal with $\langle e_3, e_3\rangle = -1$. The only non vanishing Levi-Civita products are $e_1.e_2 = e_3$ and $e_1.e_3 = e_2$. Thus the product is of Novikov.

The second example is an example of Lorentzian Novikov algebras $(\mathrm{A}, \langle\ ,\ \rangle)$ such that $[\mathrm{A_L}, \mathrm{A_L}]$ is degenerate.

Example

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat Lorentzian 2-step nilpotent Lie algebra, then $\mathfrak{g}$ endowed with the Levi-Civita product is a Lorentzian Novikov algebra. In fact, the Lie algebra $\mathfrak{g}$ is a trivial extension of the 3-dimensional Heisenberg Lie algebra $\mathcal{H}_3$. That is $\mathfrak{g} = Z_1 \oplus \mathcal{H}_3$ where $Z_1 \subset \mathfrak{z}(\mathfrak{g})$ and $\mathcal{H}_3$ is Lorentzian with degenerate center. Thus, we can find a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ such that the Lie bracket is given by $[e_1, e_2] = \lambda e_3$ where $\lambda \in \mathbb{R}^*$ and the only non vanishing scalar products on $\mathcal{H}_3$ are $\langle e_1, e_3\rangle = \langle e_2, e_2\rangle = 1$. The non vanishing Levi-Civita products are $e_1.e_1 = -\lambda e_2$ and $e_1.e_2 = \lambda e_3$ wich implies that this product is of Novikov.

Lorentzian Novikov algebras with nilpotent underlying Lie algebras

Theorem (Aubert-Medina, Tohoku Math. J. , 2003)

A nilpotent Lie group admits a flat left-invariant Lorentzian metric if and only if its Lie algebra $\mathfrak{g}$ is a double extension of an Euclidean abelian Lie algebra according to $\mu = 0$, $D = \xi$ and b_0 where $D^2 = 0$. Furthermore, $\mathfrak{g}$ is at most 3-step nilpotent.

With notations of the last theorem, let us characterize Novikov Lorentzian algebras $(\mathrm{A}, \langle\ ,\ \rangle)$ such that $\mathrm{A_L}$ is nilpotent.

Proposition

The Levi-Civita product of a flat Lorentzian nilpotent Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ is of Novikov if and only if $b_0 \in \ker \xi$.

Proof.

Since B is abelian, then the Levi-Civita product of $(B, \langle\ ,\ \rangle_B)$ is trivial. Let $a, b, c \in B$. Since $\mu = 0$ and $D = \xi$, we have,

$$[\mathrm{R}_a, \mathrm{R}_b] = [\mathrm{R}_a, \mathrm{R}_e] = [\mathrm{R}_e, \mathrm{R}_{\bar{e}}] = \mathrm{R}_{\bar{e}}\mathrm{R}_a = 0.$$

Then $\mathfrak{g}$ is of Novikov if and only if $\mathrm{R}_a\mathrm{R}_{\bar{e}} = 0$ for any $a \in B$. We have $\mathrm{R}_a\mathrm{R}_{\bar{e}}(e) = 0$ and $\langle \mathrm{R}_a\mathrm{R}_{\bar{e}}(b), e\rangle = \langle \mathrm{R}_a\mathrm{R}_{\bar{e}}(b), c\rangle = 0$. Since $\xi^2 = 0$, then

$$\langle \mathrm{R}_a\mathrm{R}_{\bar{e}}(b), \bar{e}\rangle = \langle (b.\bar{e}).a, \bar{e}\rangle = -\langle \xi(b).a, \bar{e}\rangle = \langle \xi^2(b), a\rangle = 0.$$

Thus $\mathrm{R}_a\mathrm{R}_{\bar{e}}(b) = 0$. On the other hand, we have $\langle \mathrm{R}_a\mathrm{R}_{\bar{e}}(\bar{e}), e\rangle = \langle \mathrm{R}_a\mathrm{R}_{\bar{e}}(\bar{e}), b\rangle = 0$, and

$$\langle \mathrm{R}_a\mathrm{R}_{\bar{e}}(\bar{e}), \bar{e}\rangle = \langle (\bar{e}.\bar{e}).a, \bar{e}\rangle = \langle b_0.a, \bar{e}\rangle = \langle \xi(b_0), a\rangle_B,$$

for any $a \in B$. Thus $\mathfrak{g}$ is of Novikov if and only if $b_0 \in \ker \xi$. □

Lorentzian Novikov algebras with nilpotent underlying Lie algebras

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ be a Lorentzian Novikov algebra. Then, $\mathrm{A_L}$ is nilpotent if and only if $\mathrm{A_L}$ is a double extension of an Euclidean abelian Lie algebra $(B, \langle\ ,\ \rangle_B)$ according to $\mu = 0$, $D = \xi$ and b_0 where $\xi^2 = 0$ and $b_0 \in \ker \xi$. That is, $\mathrm{A_L} = \mathbb{R}e \oplus B \oplus \mathbb{R}\bar{e}$ endowed by the non-trivial Lie brackets, for any $a, b \in B$,

$$[\bar{e}, a] = \xi(a) - \langle b_0, a\rangle_B e \text{ and } [a, b] = \langle(\xi - \xi^*)(a), b\rangle e,$$

where $\xi \in \mathrm{End}(B)$ such that $\xi^2 = 0$ and $b_0 \in \ker \xi$. The non-trivial Novikov products are

$$a.b = \langle\xi(a), b\rangle_B e,\ \ \bar{e}.\bar{e} = b_0,\ \ \bar{e}.a = -\langle b_0, a\rangle_B e \text{ and } a.\bar{e} = -\xi(a).$$

Note that, in this case, $\mathrm{A_L}$ is at most 3-step nilpotent and if $\mathrm{A_L}$ is not abelian then, the derived Lie algebra $[\mathrm{A_L}, \mathrm{A_L}]$ is degenerate.

Pseudo-Euclidean Novikov algebras with non-degenerate Lie derived ideal

Pseudo-Euclidean Novikov algebras with non-degenerate Lie derived ideal

Proposition

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ *be a pseudo-Euclidean Novikov algebra. Then*

1. $[\mathrm{A_L}, \mathrm{A_L}]$ *is a two-sided ideal.*
2. *For any* $x \in [\mathrm{A_L}, \mathrm{A_L}]^{\perp}$, R_x *is symmetric and* $\mathrm{R}_x^3 = 0$.
3. *If* $[\mathrm{A_L}, \mathrm{A_L}]$ *is non-degenerate, then* $\mathrm{R}_x = 0$ *for any* $x \in [\mathrm{A_L}, \mathrm{A_L}]^{\perp}$*. In particular,* $[\mathrm{A_L}, \mathrm{A_L}]^{\perp}$ *is abelian.*

Proof.

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Novikov Algebra.

1. From $[\mathrm{L}_x - \mathrm{ad}_x, \mathrm{L}_y - \mathrm{ad}_y]z = 0$ for any $x, y, z \in \mathrm{A}$, we deduce that

$$\begin{aligned} 0 &= [x, y].z - [\mathrm{ad}_x, \mathrm{L}_y]z - [\mathrm{L}_x, \mathrm{ad}_y]z + [[x, y], z] \\ &= [x, y].z + y.[x, z] + x.[z, y] + [y.z, x] + [y, x.z] + [[x, y], z]\,, \end{aligned}$$

hence $[x, y].z + y.[x, z] + x.[z, y] \in [\mathrm{A_L}, \mathrm{A_L}]$. We have

$$\begin{aligned} y.[x, z] + x.[z, y] &= [x, z].y + [y, [x, z]] + [z, y].x + [x, [z, y]] \\ &= (x.z).y - (z.x).y + (z.y).x - (y.z).x + [y, [x, z]] + [x, [z, y]] \\ &= (x.y - y.x).z + [y, [x, z]] + [x, [z, y]] \\ &= [x, y].z + [y, [x, z]] + [x, [z, y]]\,, \end{aligned}$$

then $[x, y].z \in [\mathrm{A_L}, \mathrm{A_L}]$. Since $z.[x, y] = [z, [x, y]] + [x, y].z$, thus $[\mathrm{A_L}, \mathrm{A_L}]$ is a two-sided ideal.

2. Let $x \in [\mathrm{A_L}, \mathrm{A_L}]^{\perp}$ and $y, z \in \mathrm{A_L}$. We have $\langle y.x, z\rangle = \langle z.x, y\rangle$, which implies that R_x is symmetric. On the other hand, (**??**) is equivalent to

$$\mathrm{R}_{u.v} - \mathrm{R}_v\mathrm{R}_u = [\mathrm{L}_u, \mathrm{R}_v],$$

for any $u, v \in \mathrm{A}$. Since $x.x = 0$ then, from the Novikov condition, we deduce that $\mathrm{R}_x\mathrm{L}_x = 0$. On the other hand, (**??**) implies that $[\mathrm{R}_x, \mathrm{L}_x] = \mathrm{R}_x^2$. Thus,

$$\mathrm{R}_x^3 = \mathrm{R}_x[\mathrm{R}_x, \mathrm{L}_x] = -\mathrm{R}_x\mathrm{L}_x\mathrm{R}_x = 0.$$

3. If $[\mathrm{A_L}, \mathrm{A_L}]$ is non-degenerate, then $\mathrm{A_L} = [\mathrm{A_L}, \mathrm{A_L}]^{\perp} \oplus [\mathrm{A_L}, \mathrm{A_L}]$. Let $x, y \in [\mathrm{A_L}, \mathrm{A_L}]^{\perp}$, we have $x.y = \frac{1}{2}[x, y] \in [\mathrm{A_L}, \mathrm{A_L}]$.

□

Pseudo-Euclidean Novikov algebras with non-degenerate Lie derived ideal

Since L_x is skew-symmetric and $[\mathrm{A_L}, \mathrm{A_L}]$ is stable by L_x, then $[\mathrm{A_L}, \mathrm{A_L}]^{\perp}$ is also stable by L_x. Thus $x.y \in [\mathrm{A_L}, \mathrm{A_L}] \cap [\mathrm{A_L}, \mathrm{A_L}]^{\perp}$ which implies that $x.y = 0$. Let $u \in [\mathrm{A_L}, \mathrm{A_L}]$. Since $[\mathrm{A_L}, \mathrm{A_L}]$ is a two-sided ideal, then $u.x \in [\mathrm{A_L}, \mathrm{A_L}]$. For any $v \in [\mathrm{A_L}, \mathrm{A_L}]$ we have $\langle u.x, v\rangle = -\langle x, u.v\rangle = 0$, thus $\mathrm{R}_x = 0$ for any $x \in [\mathrm{A_L}, \mathrm{A_L}]^{\perp}$.

In order to characterize pseudo-Euclidean Novikov algebras $(\mathrm{A}, \langle\ ,\ \rangle)$ where $[\mathrm{A_L}, \mathrm{A_L}]$ is Lorentzian, we prove first two Lemmas.

Lemma

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Novikov Algebra where $[\mathrm{A_L}, \mathrm{A_L}]$ is Lorentzian. Then $\big([\mathrm{A_L}, \mathrm{A_L}], \langle\ ,\ \rangle_{[\mathrm{A_L}, \mathrm{A_L}]}\big)$ can not be a flat Lorentzian 2-step nilpotent Lie algebra.

Proof.

Assume that $\left([\mathrm{A_L}, \mathrm{A_L}], \langle\ ,\ \rangle_{[\mathrm{A_L},\mathrm{A_L}]}\right)$ is a flat Lorentzian 2-step nilpotent Lie algebra. Then, $[\mathrm{A_L}, \mathrm{A_L}]$ must be a trivial extension of $\mathcal{H}_3$. That is $[\mathrm{A_L}, \mathrm{A_L}] = Z_1 \oplus \mathcal{H}_3$ where Z_1 is Euclidean, $Z_1 \subset \mathfrak{z}([\mathrm{A_L}, \mathrm{A_L}])$ and $\mathcal{H}_3$ is Lorentzian with degenerate center. Thus, we can find a basis $\{e, b, \bar{e}\}$ of $\mathcal{H}_3$ such that $[\bar{e}, b] = \lambda e$ where $\lambda \in \mathbb{R}^*$ and the only non vanishing scalar products on $\mathcal{H}_3$ are $\langle e, \bar{e}\rangle = \langle b, b\rangle = 1$. Let $\{e_1, \dots, e_r\}$ be an orthonormal basis of $[\mathrm{A_L}, \mathrm{A_L}]^\perp$. Since $\mathrm{R}_{e_i} = 0$, then ad_{e_i} is skew-symmetric for any $i \in \{1, \dots, r\}$. We put

$$\begin{array}{lcl} [e_i, b] & = & \alpha_i e + \beta_i \bar{e} + z_i, \\ [e_i, \bar{e}] & = & -\alpha_i b - \gamma_i \bar{e} + u_i, \\ [e_i, e] & = & \gamma_i e - \beta_i b + w_i. \end{array} \qquad \text{where } z_i, u_i, w_i \in Z_1 \text{ and } \alpha_i, \beta_i, \gamma_i \in \mathbb{R}.$$

Using (**??**), we can check that $\mathrm{L}_z = \mathrm{L}_e = \mathrm{L}_b = 0$ for any $z \in Z_1$. From

$$\langle [e_i, \bar{e}].e - e_i.(\bar{e}.e) + \bar{e}.(e_i.e), \bar{e}\rangle = 0,$$

we deduce that $-\gamma_i\langle \bar{e}.e, \bar{e}\rangle + \langle e_i.\bar{e}, \bar{e}.e\rangle - \langle e_i.e, \bar{e}.\bar{e}\rangle = 0$. Since $\mathrm{L}_{e_i} = \mathrm{ad}_{e_i}$ and $\langle \bar{e}.e, \bar{e}\rangle = 0$ then $\langle -\alpha_i b - \gamma_i \bar{e} + u_i, \bar{e}.e\rangle - \langle \gamma_i e - \beta_i b + w_i, \bar{e}.\bar{e}\rangle = 0$, which implies that $\beta_i\langle \bar{e}.\bar{e}, b\rangle = 0$, thus $\lambda\beta_i = 0$ for any $i \in \{1, \dots, r\}$. In the same way, from

$$\langle [e_i, \bar{e}].b - e_i.(\bar{e}.b) + \bar{e}.(e_i.b), \bar{e}\rangle = 0,$$

we show that $\lambda\gamma_i = 0$ for any $i \in \{1, \dots, r\}$. Since $\lambda \neq 0$ then $\beta_i = \gamma_i = 0$ for any $i \in \{1, \dots, r\}$, which implies that $\bar{e} \notin [\mathrm{A_L}, \mathrm{A_L}]$. This gives a contradiction, and completes the proof. □

Pseudo-Euclidean Novikov algebras with non-degenerate Lie derived ideal

Lemma

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ be a pseudo-Euclidean Novikov algebra where $[\mathrm{A_L}, \mathrm{A_L}]$ is Lorentzian. Then $\left([\mathrm{A_L}, \mathrm{A_L}], \langle\ ,\ \rangle_{[\mathrm{A_L},\mathrm{A_L}]}\right)$ can not be a flat Lorentzian 3-step nilpotent Lie algebra.

Proof.

Assume that $\left([\mathrm{A_L}, \mathrm{A_L}], \langle\ ,\ \rangle_{[\mathrm{A_L},\mathrm{A_L}]}\right)$ is a flat Lorentzian 3-step nilpotent Lie algebra. According to Theorem **??**, $\left([\mathrm{A_L}, \mathrm{A_L}], \langle\ ,\ \rangle_{[\mathrm{A_L},\mathrm{A_L}]}\right)$ must be a double extension of an Euclidean abelian Lie algebra B according to $\mu = 0$, $D = \xi$ and b_0 where $D^2 = 0$ and $D \neq 0$. We have obviously, $\mathrm{Im}\ D \subset \ker D$. We put $rank(D) = r \geq 1$, $B = \mathrm{Im}\ D \oplus (\mathrm{Im}\ D)^{\perp}$ and let $\{b_1, \dots, b_r\}$ (resp. $\{b_{r+1}, \dots, b_p\}$) be an orthonormal basis of $\mathrm{Im}\ D$ (resp. of $(\mathrm{Im}\ D)^{\perp}$). Note that, $p - r \geq r$. Let $D = (d_{ij})_{1 \leq i,j \leq p}$ be the matrix of D in the basis $\{b_1, \dots, b_p\}$. We have $d_{ij} = 0$ for all $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, r\}$, and $d_{ij} = 0$ for all $i, j \in \{r+1, \dots, p\}$. Let $\overline{D}$ be the submatrix of D formed by d_{ij} for $i \in \{1, \dots, r\}$ and $j \in \{r+1, \dots, p\}$. Then we have $rank(\overline{D}) = r$. We Put $\mathrm{A_L} = [\mathrm{A_L}, \mathrm{A_L}]^{\perp} \oplus [\mathrm{A_L}, \mathrm{A_L}]$ where $[\mathrm{A_L}, \mathrm{A_L}] = \mathbb{R}e \oplus B \oplus \mathbb{R}\bar{e}$. According to (**??**), the only non vanishing Lie brackets in $[\mathrm{A_L}, \mathrm{A_L}]$ are given by

$$\begin{array}{rcl} [\bar{e}, b_i] & = & \lambda_i e, \quad \text{for any } i \in \{1, \dots, r\}, \\ [\bar{e}, b_i] & = & \sum_{k=1}^{r} d_{ki} b_k + \lambda_i e, \quad \text{for any } i \in \{r+1, \dots, p\}, \\ [b_i, b_j] & = & -d_{ij} e, \quad \text{for any } i \in \{1, \dots, r\} \text{ and } j \in \{r+1, \dots, p\}. \end{array}$$

Let $\{e_1, \dots, e_q\}$ be an orthonormal basis of $[\mathrm{A_L}, \mathrm{A_L}]^{\perp}$. Since $\mathrm{R}_{e_s} = 0$ then ad_{e_s} is skew-symmetric for any $s \in \{1, \dots, q\}$. Then we can put for any $s \in \{1, \dots, q\}$

$$\begin{array}{lll} [e_s, e] & = & xe + \sum_{i=1}^{p} x_i b_i, \\ [e_s, b_i] & = & \alpha_i e + \sum_{k=1}^{p} a_{ki} b_k - x_i \bar{e}, \\ [e_s, \bar{e}] & = & -\sum_{i=1}^{p} \alpha_i b_i - x\bar{e}, \end{array}$$

where $x, x_i, \alpha_i \in \mathbb{R}$ for any $i \in \{1, \dots, p\}$, and $(a_{ij})_{1 \le i,j \le p}$ is a skew-symmetric matrix. Let $j \in \{r+1, \dots, p\}$, from the Jacobi identity $\big[[e_s, e], b_j\big] + \big[[e, b_j], e_s\big] + \big[[b_j, e_s], e\big] = 0$ we deduce that $\sum_{i=1}^{r} x_i [b_i, b_j] = 0$, which implies that $\sum_{i=1}^{r} d_{ij} x_i = 0$ for any $j \in \{r+1, \dots, p\}$. Since $rank(\overline{D}) = r$, thus $x_i = 0$ for any $i \in \{1, \dots, r\}$.
Let $i \in \{1, \dots, r\}$ and $j \in \{r+1, \dots, p\}$. Let $\big[[e_s, b_i], b_j\big]^B$ (resp. $\big[[b_i, b_j], e_s\big]^B$, $\big[[b_j, e_s], b_i\big]^B$) be the compenent of $\big[[e_s, b_i], b_j\big]$ (resp. $\big[[b_i, b_j], e_s\big]$, $\big[[b_j, e_s], b_i\big]$) with respect to B. Since $x_i = 0$ for any $i \in \{1, \dots, r\}$, then $\big[[e_s, b_i], b_j\big]^B = \big[[b_j, e_s], b_i\big]^B = 0$, and $\big[[b_i, b_j], e_s\big]^B = d_{ij} \sum_{k=r+1}^{p} x_k b_k = 0$. Since there exists a non nul d_{ij}, then $x_i = 0$ for any $i \in \{r+1, \dots, p\}$. Let us show now that $x = 0$. Let $i \in \{r+1, \dots, p\}$. We have $\langle [e_s, b_i].\bar{e} - e_s.(b_i.\bar{e}) + b_i.(e_s.\bar{e}), b_1 \rangle = 0$, then

$$\langle [e_s, b_i].\bar{e}, b_1 \rangle + \langle e_s.b_1, b_i.\bar{e} \rangle = \langle e_s.\bar{e}, b_i.b_1 \rangle.$$

Since the product is of Novikov and $\mathrm{L}_{e_s} = \mathrm{ad}_{e_s}$, then

$$\langle [e_s, b_i].\bar{e}, b_1 \rangle = \langle (e_s.b_i).\bar{e}, b_1 \rangle = -\langle (e_s.b_i).b_1, \bar{e} \rangle = -\langle (e_s.b_1).b_i, \bar{e} \rangle = -\langle [e_s.b_1].b_i, \bar{e} \rangle.$$

According to (**??**), we can check that $\mathrm{L}_e = \mathrm{L}_{b_j} = 0$ for any $j \in \{1, \dots, r\}$. Then $\langle [e_s, b_1].b_i, \bar{e} \rangle = \sum_{k=r+1}^{p} a_{k1} \langle b_k.b_i, \bar{e} \rangle = 0$ (because $i, k \in \{r+1, \dots, p\}$). Thus $\langle [e_s, b_1], b_i.\bar{e} \rangle = \langle [e_s, \bar{e}], b_i.b_1 \rangle$, which implies that $\sum_{k=1}^{r} a_{k1} \langle b_k, b_i.\bar{e} \rangle = -x \langle b_i.b_1, \bar{e} \rangle$. Since $(a_{ij})_{1 \le i,j \le p}$ is a skew-symmetric matrix, then we deduce the linear system

$$d_{1i} x + \sum_{k=2}^{r} d_{ki} a_{1k} = 0, \text{ for any } i \in \{r+1, \dots, p\}.$$

Since $rank(\overline{D}) = r$, then $x = a_{1k} = 0$ for any $k \in \{2, \dots, r\}$, and hence $\bar{e} \notin [\mathrm{A_L}, \mathrm{A_L}]$. This implies a contradiction, and completes the proof of the Lemma.

Pseudo-Euclidean Novikov algebras with non-degenerate Lie derived ideal

Theorem

$(\mathrm{A}, \langle\ ,\ \rangle)$ is a pseudo-Euclidean Novikov Algebra such that $[\mathrm{A_L}, \mathrm{A_L}]$ is Lorentzian if and only if $\mathrm{A_L} = [\mathrm{A_L}, \mathrm{A_L}]^{\perp} \oplus [\mathrm{A_L}, \mathrm{A_L}]$, where $[\mathrm{A_L}, \mathrm{A_L}]^{\perp}$ and $[\mathrm{A_L}, \mathrm{A_L}]$ are abelian, $[\mathrm{A_L}, \mathrm{A_L}]$ is Lorentzian and ad_x is skew-symmetric for any $x \in [\mathrm{A_L}, \mathrm{A_L}]^{\perp}$.

Proof.

$\mathrm{A_L}$ must be solvable, then $[\mathrm{A_L}, \mathrm{A_L}]$ is nilpotent. Since $[\mathrm{A_L}, \mathrm{A_L}]$ is a two-sided ideal, then $\left([\mathrm{A_L}, \mathrm{A_L}], \langle\ ,\ \rangle_{[\mathrm{A_L},\mathrm{A_L}]}\right)$ is a flat Lorentzian nilpotent Lie algebra. According to Theorem of Aubert and Medina, $[\mathrm{A_L}, \mathrm{A_L}]$ is at most 3-step nilpotent. By applying Lemma 1 and Lemma 2, we deduce that $[\mathrm{A_L}, \mathrm{A_L}]$ must be abelian. On the other hand, we have $\mathrm{R}_x = 0$ for any $x \in [\mathrm{A_L}, \mathrm{A_L}]^{\perp}$. Then $[\mathrm{A_L}, \mathrm{A_L}]^{\perp}$ is abelian and $\mathrm{ad}_x = \mathrm{L}_x$ is skew-symmetric for any $x \in [\mathrm{A_L}, \mathrm{A_L}]^{\perp}$. □

Pseudo-Euclidean Novikov algebras with non-degenerate Lie derived ideal

Remark

The simplest example of the last Theorem, is the Lie group of rigid motions of Minkowski plane $\mathrm{E}(1,1)$. *Note that this Lie group admits no flat left-invariant Riemannian metric. Indeed, if not,* $\mathrm{E}(1,1)$ *will be isomorphic to the Euclidean group* $\mathrm{E}(2)$.

Corollary

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ *be a pseudo-Euclidean Novikov algebra such that* $[\mathrm{A_L}, \mathrm{A_L}]$ *is Lorentzian. Then* $\mathrm{A_L}$ *is 2-solvable and unimodular.*

Proof.

According to the last Theorem, we have $\mathrm{tr}(\mathrm{ad}_x) = 0$ for any $x \in \mathrm{A_L}$, which implies that $\mathrm{A_L}$ is unimodular. Since $[\mathrm{A_L}, \mathrm{A_L}]$ is abelian, then $\mathrm{A_L}$ is 2-solvable. □

Pseudo-Euclidean Novikov algebras with non-degenerate Lie derived ideal

Remark

It is well-known that a flat pseudo-Riemannian connected Lie group is geodesically complete if and only if it is unimodular. This is also equivalent to the fact that the associated left-symmetric structure is transitive. That is all right multiplications are nilpotent. Thus, according to Corollary 3.1, if $(\mathrm{A}, \langle\ ,\ \rangle)$ *is a pseudo-Euclidean Novikov algebra such that* $[\mathrm{A_L}, \mathrm{A_L}]$ *is Lorentzian, then* A *is transitive and the metric is complete.*

Theorem

$(\mathrm{A}, \langle\ ,\ \rangle)$ *is a Lorentzian Novikov algebra such that* $[\mathrm{A_L}, \mathrm{A_L}]$ *is non-degenerate if and only if* $\mathrm{A_L} = [\mathrm{A_L}, \mathrm{A_L}]^{\perp} \oplus [\mathrm{A_L}, \mathrm{A_L}]$, *where* $[\mathrm{A_L}, \mathrm{A_L}]^{\perp}$ *and* $[\mathrm{A_L}, \mathrm{A_L}]$ *are abelian, and* ad_x *is skew-symmetric for any* $x \in [\mathrm{A_L}, \mathrm{A_L}]^{\perp}$.

Proof.

We have two cases: if $[\mathrm{A_L}, \mathrm{A_L}]$ is Euclidean then we can apply Theorem (L, Proc. A.M.S (2020)). If $[\mathrm{A_L}, \mathrm{A_L}]$ is Lorentzian then we conclude by the last Theorem. □

Pseudo-Euclidean Novikov algebras with degenerate Lie derived ideal

Pseudo-Euclidean Novikov algebras with degenerate Lie derived ideal

Proposition

Let $(\mathrm{A}, \langle \ , \ \rangle)$ *be a Lorentzian Novikov algebra. Then* $\mathrm{L}_e = \mathrm{R}_e = \mathrm{ad}_e = 0$ *for any* $e \in [\mathrm{A_L}, \mathrm{A_L}] \cap [\mathrm{A_L}, \mathrm{A_L}]^\perp$.

Proof.

The result is trivial if $[\mathrm{A_L}, \mathrm{A_L}]$ is non-degenerate. Assume that $[\mathrm{A_L}, \mathrm{A_L}]$ is degenerate, and put $[\mathrm{A_L}, \mathrm{A_L}] \cap [\mathrm{A_L}, \mathrm{A_L}]^\perp = \mathbb{R}e$. Thus $\mathfrak{g}$ splits as

$$\mathrm{A_L} = \mathrm{U} \oplus \mathbb{R}e \oplus \mathrm{W} \oplus \mathbb{R}\bar{e},$$

where $[\mathrm{A_L}, \mathrm{A_L}]^\perp = \mathrm{U} \oplus \mathbb{R}e$, $[\mathrm{A_L}, \mathrm{A_L}] = \mathbb{R}e \oplus \mathrm{W}$ and U and W are Euclidean. Let $\{u_1, \ldots, u_r\}$ (resp. $\{w_1, \ldots, w_q\}$) be an orthonormal basis of U (resp. W). For all $x, y \in [\mathrm{A_L}, \mathrm{A_L}]^\perp$, we have $x.y = \frac{1}{2}[x, y] \in [\mathrm{A_L}, \mathrm{A_L}] \cap [\mathrm{A_L}, \mathrm{A_L}]^\perp$. Then we can put $e.u_i = -u_i.e = \frac{1}{2}\alpha_i e$, where $\alpha_i \in \mathbb{R}$. We have $\mathrm{R}^k_{u_i}(e) = \left(\frac{1}{2}\alpha_i\right)^k e$, for any $k \in \mathbb{N}^*$. Since $u_i \in [\mathrm{A_L}, \mathrm{A_L}]^\perp$ then R_{u_i} is nilpotent and hence $\alpha_i = 0$ for any $i \in \{1, \ldots, r\}$. Let $j \in \{1, \ldots, q\}$ and $x \in \mathrm{A_L}$. From

$$\langle \mathrm{R}_e(w_j), x\rangle = \langle w_j.e, x\rangle = -\langle e, w_j.x\rangle,$$

and the fact that $w_j.x \in [\mathrm{A_L}, \mathrm{A_L}]$, we deduce that $\mathrm{R}_e(w_j) = 0$ for any $j \in \{1, \ldots, q\}$. Thus $\mathrm{R}_e(y) = 0$ for any $y \in [\mathrm{A_L}, \mathrm{A_L}] + [\mathrm{A_L}, \mathrm{A_L}]^\perp$. Since R_e is symmetric, then $\mathrm{R}_e(\bar{e}) = \lambda e$. □

We have

$$\langle \mathrm{R}_{\bar{e}}\mathrm{R}_e(\bar{e}), \bar{e}\rangle = \langle (\bar{e}.e).\bar{e}, \bar{e}\rangle = \lambda\langle e.\bar{e}, \bar{e}\rangle = 0,$$

and

$$\langle \mathrm{R}_e\mathrm{R}_{\bar{e}}(\bar{e}), \bar{e}\rangle = \langle (\bar{e}.\bar{e}), \bar{e}.e\rangle = \lambda\langle \bar{e}.\bar{e}, e\rangle = -\lambda\langle \bar{e}, \bar{e}.e\rangle = -\lambda^2,$$

thus $\lambda = 0$ and $\mathrm{R}_e = 0$. Then $\mathrm{ad}_e = \mathrm{L}_e$ is skew-symmetric. Recall that $e.u_i = [e, u_i] = 0$ for any $i \in \{1, \ldots, r\}$. Let $\overline{\mathrm{ad}_e}$ be the restriction of the skew-symmetric endomorphism ad_e to $[\mathrm{A_L}, \mathrm{A_L}]$. The matrix of $\overline{\mathrm{ad}_e}$ in the basis $\{e, w_1, \ldots, w_q\}$ has the form

$$\begin{pmatrix} 0 & \beta \\ 0 & M \end{pmatrix}$$

where $\beta = (\beta_1, \ldots, \beta_q) \in \mathbb{R}^q$, and M is a $q \times q$-skew-symmetric matrix. Since $\mathrm{A_L}$ is solvable then $[\mathrm{A_L}, \mathrm{A_L}]$ is nilpotent.

Thus M is a skew-symmetric nilpotent matrix which implies that $M = 0$. Then, we have $\mathrm{ad}_{w_i}(e) = [w_i, e] = -\beta_i e$ and $\mathrm{ad}^k_{w_i}(e) = (-\beta_i)^k e$ for any $k \in \mathbb{N}^*$. Thus $\beta_i = 0$ for any $i \in \{1, \ldots, q\}$. Now, from $\langle [e, \bar{e}], y\rangle = -\langle \bar{e}, [e, y]\rangle = 0$ for any $y \in [\mathrm{A_L}, \mathrm{A_L}] + [\mathrm{A_L}, \mathrm{A_L}]^\perp$, and $\langle [e, \bar{e}], \bar{e}\rangle = 0$ we deduce that $\mathrm{L}_e = \mathrm{R}_e = \mathrm{ad}_e = 0$, as desired.

Theorem

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ be a Lorentzian Novikov algebra where $[\mathrm{A_L}, \mathrm{A_L}]$ is degenerate. Then $(\mathrm{A_L}, \langle\ ,\ \rangle)$ is obtained by the double extension process from a flat Euclidean Lie algebra according to $\mu = 0$, D, ξ and b_0 where $\xi^2 = 0$. Furthermore, $\mathrm{A_L}$ is unimodular.

Proof.

Put $[\mathrm{A_L}, \mathrm{A_L}] \cap [\mathrm{A_L}, \mathrm{A_L}]^{\perp} = \mathbb{R}e$. We have $\mathrm{L}_e = \mathrm{R}_e = 0$. Then $\mathrm{I} = \mathbb{R}e$ is a totally isotropic one dimensional two-sided ideal, and $\mathrm{I}^{\perp}$ is also a two-sided ideal. Thus $(\mathrm{A_L}, \langle\ ,\ \rangle)$ is a double extension of a flat Euclidean Lie algebra $(B, \langle\ ,\ \rangle_B)$ according to μ, D, ξ and b_0. From

$$\langle \mathrm{R}_e \mathrm{R}_{\bar{e}}(\bar{e}), \bar{e}\rangle = \langle(\bar{e}.\bar{e}).e, \bar{e}\rangle = -\mu\langle \bar{e}.e, \bar{e}\rangle = -\mu^2,$$

and

$$\langle \mathrm{R}_{\bar{e}} \mathrm{R}_e(\bar{e}), \bar{e}\rangle = \langle(\bar{e}.e).\bar{e}, \bar{e}\rangle = \mu\langle e.\bar{e}, \bar{e}\rangle = 0,$$

we deduce that $\mu = 0$. According to (4), we have for any $a, b \in B$

$$\langle \mathrm{R}_{\bar{e}} \mathrm{R}_a(b), \bar{e}\rangle = \langle(b.a).\bar{e}, \bar{e}\rangle = 0,$$

and

$$\langle \mathrm{R}_a \mathrm{R}_{\bar{e}}(b), \bar{e}\rangle = \langle(b.\bar{e}).a, \bar{e}\rangle = -\langle \xi(b).a, \bar{e}\rangle = \langle \xi^2(b), a\rangle,$$

thus $\xi^2 = 0$. Let $a, b \in B$ and $[a, b]_B$ the Lie bracket in B, and let $\overline{\mathrm{ad}_a}$ be the endomorphism of B defined by $\overline{\mathrm{ad}_a}(b) = [a, b]_B$. We have $\mathrm{tr}(\mathrm{ad}_a) = \mathrm{tr}(\overline{\mathrm{ad}_a})$ for any $a \in B$. Since B is unimodular, then $\mathrm{tr}(\mathrm{ad}_a) = 0$ for any $a \in B$. On the other hand, we have $\mathrm{tr}(\mathrm{ad}_{\bar{e}}) = \mathrm{tr}(D)$. Since $D - \xi$ is skew-symmetric and ξ is nilpotent, then $\mathrm{tr}(\mathrm{ad}_{\bar{e}}) = 0$. Thus $\mathrm{A_L}$ is unimodular. □

Pseudo-Euclidean Novikov algebras with degenerate Lie derived ideal

Let us give some applications of Theorem 4.1 in low dimensions. Note first that, since $\mathrm{A_L}$ must be unimodular, then there exists no 2-dimensional Lorentzian Novikov algebra such that $[\mathrm{A_L}, \mathrm{A_L}]$ is degenerate. Note also that if $\mathrm{A_L}$ is abelian then $[\mathrm{A_L}, \mathrm{A_L}]$ is obviously non-degenerate.

Proposition

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ *be a Lorentzian Novikov algebra where* $[\mathrm{A_L}, \mathrm{A_L}]$ *is degenerate. Then*

1. *If* $\dim \mathrm{A} = 3$, *then* $\mathrm{A_L}$ *is isomorphic to* $\mathcal{H}_3$.
2. *If* $\dim \mathrm{A} = 4$, *then* $\mathrm{A_L}$ *is isomorphic to one of these Lie algebras:*
   - $\mathbb{R} \oplus \mathcal{H}_3$.
   - $\mathfrak{g}_4$, *the filiform Lie algebra of dimension 4.*

Proof.

Note first that if B is abelian, then $(\mu = 0, D, \xi, b_0)$ is admissible if and only if $D - \xi$ is skew-symmetric and $[D, \xi] = \xi^2$. In this case, the product is of Novikov if and only if

$$\begin{cases} D - \xi \text{ is skew-symmetric,} \\ \xi^2 = 0, \\ D\xi = \xi D = 0, \\ b_0 \in \ker \xi. \end{cases} \quad (5)$$

If $\dim A = 3$ then $\dim B = 1$, which implies that $D = \xi = 0$. Put $b_0 = \lambda b$ where $B = \mathbb{R}b$ with $\langle b, b \rangle = 1$. Thus, $\mathrm{A_L} = \mathbb{R}e \oplus \mathbb{R}b \oplus \mathbb{R}\bar{e}$ where the only non vanishing Lie bracket is $[\bar{e}, b] = \lambda e$, with $\lambda \neq 0$. Thus $\mathrm{A_L}$ is isomorphic to $\mathcal{H}_3$. If $\dim \mathrm{A} = 4$ then $\dim B = 2$ and we can find an orthormal basis $\{b_1, b_2\}$ of B such that D, ξ and b_0 have the forms

$$\xi = D = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \text{ and } b_0 = \lambda b_1, \quad \text{or} \quad \xi = 0,\ D = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \text{ and } b_0 = \lambda_1 b_1 + \lambda_2 b_2.$$

In the first case, the Lie brackets are given by $[\bar{e}, b_1] = \lambda e$, $[\bar{e}, b_2] = ab_1$ and $[b_1, b_2] = -ae$, where $(\lambda, a) \neq (0, 0)$. If $a = 0$ then $\mathrm{A_L} \cong \mathbb{R} \oplus \mathcal{H}_3$. If $a \neq 0$ then it's easy to find a basis where the Lie brackets are given by $[x_1, x_2] = x_3$ and $[x_1, x_3] = x_4$, which implies that $\mathrm{A_L} \cong \mathfrak{g}_4$. In the second case, one can check that $[\mathrm{A_L}, \mathrm{A_L}]$ is degenerate iff $\lambda = 0$ and $(\lambda_1, \lambda_2) \neq (0, 0)$, which also implies that $\mathrm{A_L} \cong \mathbb{R} \oplus \mathcal{H}_3$. □

Pseudo-Euclidean Novikov algebras with degenerate Lie derived ideal

Remark

All Lie algebras obtained in Proposition 4.2 are nilpotent. This isn't the case in general. For example, if B *is abelian and* $\dim B = 3$, *then* ξ, D *and* b_0 *given by*

$$\xi = 0, \;\; D = \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } b_0 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3,$$

verifies (5), *where* $\lambda, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^*$. *One can show that the Lie algebra obtained in this case, is isomorphic to the non nilpotent Lie algebra defined by the only non vanishing Lie brackets* $[x_1, x_2] = x_3$, $[x_1, x_3] = -x_2$ *and* $[x_1, x_4] = x_5$.

Lorentzian Novikov algebras

Theorem

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ *be a Lorentzian Novikov algebra. Then*

- $\mathrm{A_L}$ *is unimodular.*
- *The restriction of the product to* $[\mathrm{A_L}, \mathrm{A_L}]^\perp$ *is trivial. In particular* $[\mathrm{A_L}, \mathrm{A_L}]^\perp$ *is abelian.*

Proof.

The first point is a consequence of Theorem 3.2 and Theorem 4.1. Let us show the second point. If $[\mathrm{A_L}, \mathrm{A_L}]$ is non-degenerate, then the result follows from Proposition 3.1. Assume that $[\mathrm{A_L}, \mathrm{A_L}]$ is degenerate and put $[\mathrm{A_L}, \mathrm{A_L}] \cap [\mathrm{A_L}, \mathrm{A_L}]^\perp = \mathbb{R}e$ with $\mathrm{L}_e = \mathrm{R}_e = 0$. We have

$$\mathrm{A_L} = \mathrm{U} \oplus \mathbb{R}e \oplus \mathrm{W} \oplus \mathbb{R}\bar{e},$$

where $[\mathrm{A_L}, \mathrm{A_L}]^\perp = \mathrm{U} \oplus \mathbb{R}e$, $[\mathrm{A_L}, \mathrm{A_L}] = \mathbb{R}e \oplus \mathrm{W}$ and U and W are Euclidean. Let $\{u_1, \ldots, u_r\}$ (resp. $\{w_1, \ldots, w_q\}$) be an orthonormal basis of U (resp. W). Note that R_{u_i} is symmetric for any $i \in \{1, \ldots, r\}$. Let $i, j \in \{1, \ldots, r\}$ and $k \in \{1, \ldots, q\}$. We have $[u_i, u_j] = 2u_i.u_j \in \mathbb{R}e$. Then we can put $[u_i, u_j] = 2u_i.u_j = a_{ij}e$, where $a_{ij} \in \mathbb{R}$. Since $\bar{e}.w_k \in [\mathrm{A_L}, \mathrm{A_L}]$, then we have

$$\langle \bar{e}.u_i, w_k \rangle = -\langle u_i, \bar{e}.w_k \rangle = 0, \ \langle \bar{e}.u_i, u_j \rangle = -\frac{1}{2}a_{ij} \text{ and } \langle \bar{e}.u_i, e \rangle = 0.$$

Thus $\bar{e}.u_i = -\frac{1}{2}\sum_{j=1}^{r} a_{ij}u_j + \alpha e$, where $\alpha \in \mathbb{R}$. On the other hand, we have

$$\langle u_i.\bar{e}, \bar{e}\rangle = \langle u_i.\bar{e}, e\rangle = 0, \text{ and } \langle u_i.\bar{e}, u_j\rangle = -\frac{1}{2}a_{ij}.$$

Thus $u_i.\bar{e} = -\frac{1}{2}\sum_{j=1}^{r} a_{ij}u_j + x_0$, where $x_0 \in \mathrm{W}$. Since the product is of Novikov, then $\langle \mathrm{R}_{u_i}\mathrm{R}_{\bar{e}}(u_i), \bar{e}\rangle = \langle \mathrm{R}_{\bar{e}}\mathrm{R}_{u_i}(u_i), \bar{e}\rangle$. But

$$\langle \mathrm{R}_{\bar{e}}\mathrm{R}_{u_i}(u_i), \bar{e}\rangle = \langle \mathrm{L}_{u_i.u_i}(\bar{e}), \bar{e}\rangle = 0,$$

and

$$\langle \mathrm{R}_{u_i}\mathrm{R}_{\bar{e}}(u_i), \bar{e}\rangle = \langle \mathrm{R}_{u_i}(u_i.\bar{e}), \bar{e}\rangle = \langle u_i.\bar{e}, \mathrm{R}_{u_i}(\bar{e})\rangle = \langle u_i.\bar{e}, \bar{e}.u_i\rangle.$$

Thus $\langle u_i.\bar{e}, \bar{e}.u_i\rangle = 0$. That is $\langle -\frac{1}{2}\sum_{j=1}^{r} a_{ij}u_j + x_0, -\frac{1}{2}\sum_{j=1}^{r} a_{ij}u_j + \alpha e\rangle = 0$, which implies that $\sum_{j=1}^{r} a_{ij}^2 = 0$, and the induced product on $[\mathrm{A_L}, \mathrm{A_L}]^{\perp}$ is trivial, as desired.

Remark

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ be a Lorentzian Novikov algebra. Then, according to Theorem 4.2, A is transitive and the metric is complete.

Pseudo-Euclidean Novikov algebras of signature $(2, n-2)$ with 2-step nilpotent underlying Lie algebras

Pseudo-Euclidean Novikov algebras of signature $(2, n-2)$ with 2-step nilpotent underlying Lie algebras

In this last paragraph, we study pseudo-Euclidean Novikov algebra $(\mathrm{A}, \langle\ ,\ \rangle)$ of signature $(2, n-2)$ where $\mathrm{A_L}$ is 2-step nilpotent. We have $1 \leq \dim \mathfrak{z}(\mathrm{A_L}) \cap \mathfrak{z}(\mathrm{A_L})^\perp \leq 2$.

Proposition

$(\mathrm{A}, \langle\ ,\ \rangle)$ is a pseudo-Euclidean Novikov algebra of signature $(2, n-2)$ such that $\mathrm{A_L}$ is 2-step nilpotent and $\dim \mathfrak{z}(\mathrm{A_L}) \cap \mathfrak{z}(\mathrm{A_L})^\perp = 1$ if and only if $\mathrm{A_L}$ is a trivial extension of $\mathcal{H}_3$.

Proof.

It is known that $\mathrm{A_L}$ must be a trivial extension of $\mathcal{H}_3$ or L_6^4. Assume that $\mathrm{A_L}$ is a trivial extension of L_6^4. We have $\mathrm{R}_{x_1}\mathrm{R}_{x_2}(x_1) = -\frac{1}{3bd}x_3$, and $\mathrm{R}_{x_2}\mathrm{R}_{x_1}(x_1) = -\frac{c}{2bd}x_6$, where $bd \neq 0$. Thus $\mathrm{A_L}$ can not be a trivial extension of L_6^4. □

Pseudo-Euclidean Novikov algebras of signature $(2, n-2)$ with 2-step nilpotent underlying Lie algebras

Proposition

$(\mathrm{A}, \langle\ ,\ \rangle)$ *is a pseudo-Euclidean Novikov algebra of signature* $(2, n-2)$ *such that* $\mathrm{A_L}$ *is 2-step nilpotent and* $\dim \mathfrak{z}(\mathrm{A_L}) \cap \mathfrak{z}(\mathrm{A_L})^\perp = 2$ *if and only if*

$$\mathrm{A_L} = Z_1 \oplus \mathrm{span}\{e_1, e_2\} \oplus \mathrm{B} \oplus \mathrm{span}\{\bar{e}_1, \bar{e}_2\},$$

with the only non vanishing Lie brackets $[\bar{e}_1, \bar{e}_2] = \lambda_1 e_1 + \lambda_2 e_2$, $[\bar{e}_1, b_i] = \alpha_i e_1 + \beta_i e_2$ *and* $[\bar{e}_2, b_i] = \gamma_i e_1 + \delta_i e_2$, *where* $\{b_1, \ldots, b_r\}$ *is an orthonormal basis of* B, $\lambda_1, \lambda_2, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$ *verifiying*

$$\sum_{i=1}^{r}(\beta_i + \gamma_i)^2 = 4\sum_{i=1}^{r}\alpha_i\delta_i,\ \sum_{i=1}^{r}\gamma_i^2 = \sum_{i=1}^{r}\beta_i^2,\ \sum_{i=1}^{r}\alpha_i\beta_i = \sum_{i=1}^{r}\alpha_i\gamma_i \text{ and } \sum_{i=1}^{r}\delta_i\beta_i = \sum_{i=1}^{r}\delta_i\gamma_i. \tag{6}$$

Proof.

According to Theorem 6.2 of [?], $\mathrm{A_L} = Z_1 \oplus \mathrm{span}\{e_1, e_2\} \oplus \mathrm{B} \oplus \mathrm{span}\{\bar{e}_1, \bar{e}_2\}$ with the Lie brackets $[\bar{e}_1, \bar{e}_2] = z_0$, $[\bar{e}_1, b_i] = \alpha_i e_1 + \beta_i e_2$ and $[\bar{e}_2, b_i] = \gamma_i e_1 + \delta_i e_2$, where $\{b_1, \ldots, b_r\}$ is an orthonormal basis of B and $3\langle z_0, z_0\rangle = \sum_{i=1}^{r}(\gamma_i + \beta_i)^2 - 4\alpha_i\delta_i$. Let $\{z_1, \ldots, z_q\}$ be an orthonormal basis of Z_1 and put $z_0 = \sum_{i=1}^{q} x_i z_i + \lambda_1 e_1 + \lambda_2 e_2$. Then the only non vanishing Levi-Civita products are $\mathrm{R}_{z_i}(\bar{e}_1) = -\frac{1}{2}x_i e_2$, $\mathrm{R}_{z_i}(\bar{e}_2) = \frac{1}{2}x_i e_1$, $\mathrm{R}_{b_i}(\bar{e}_1) = \alpha_i e_1 + \frac{1}{2}(\beta_i + \gamma_i)e_2$, $\mathrm{R}_{b_i}(\bar{e}_2) = \frac{1}{2}(\beta_i + \gamma_i)e_1 + \delta_i e_2$, $\mathrm{R}_{\bar{e}_1}(z_i) = -\frac{1}{2}x_i e_2$, $\mathrm{R}_{\bar{e}_1}(b_i) = \frac{\gamma_i - \beta_i}{2} e_2$, $\mathrm{R}_{\bar{e}_1}(\bar{e}_1) = -\sum_{i=1}^{r}\alpha_i b_i - \lambda_1 e_2$, $\mathrm{R}_{\bar{e}_1}(\bar{e}_2) = -\frac{1}{2}\sum_{i=1}^{q} x_i z_i - \lambda_2 e_2 - \frac{1}{2}\sum_{i=1}^{r}(\beta_i + \gamma_i)b_i$, $\mathrm{R}_{\bar{e}_2}(z_i) = \frac{1}{2}x_i e_1$, $\mathrm{R}_{\bar{e}_2}(b_i) = \frac{\beta_i - \gamma_i}{2} e_1$, $\mathrm{R}_{\bar{e}_2}(\bar{e}_1) = \frac{1}{2}\sum_{i=1}^{q} x_i z_i + \lambda_1 e_1 - \frac{1}{2}\sum_{i=1}^{r}(\beta_i + \gamma_i)b_i$, $\mathrm{R}_{\bar{e}_2}(\bar{e}_2) = -\sum_{i=1}^{r}\delta_i b_i + \lambda_2 e_1$. Thus $(\mathrm{A}, \langle\ ,\ \rangle)$ is a pseudo-Euclidean Novikov algebra if and only if $\mathrm{R}_{\bar{e}_1}\mathrm{R}_{\bar{e}_2} = \mathrm{R}_{\bar{e}_2}\mathrm{R}_{\bar{e}_1}$, which is equivalent to the equalities (6) and $x_i = 0$ for any $i \in \{1, \ldots, q\}$. This completes the proof. □

Example

If we take $\alpha_i = \delta_i = \lambda_1 = \lambda_2 = 0$ and $\beta_i = -\gamma_i$ then

$$\mathrm{A_L} = Z_1 \oplus \mathrm{span}\{e_1, e_2\} \oplus \mathrm{B} \oplus \mathrm{span}\{\bar{e}_1, \bar{e}_2\},$$

with the Lie brackets $[\bar{e}_1, b_i] = e_1$ and $[\bar{e}_2, b_i] = -e_2$.

Corollary

Let $(\mathrm{A}, \langle\ ,\ \rangle)$ *be a pseudo-Euclidean Novikov algebra of signature* $(2, n-2)$ *such that* $\mathrm{A_L}$ *is 2-step nilpotent. Then* $\dim[\mathrm{A_L}, \mathrm{A_L}] \leq 2$ *and* $[\mathrm{A_L}, \mathrm{A_L}]$ *is totally isotropic.*

Many Thanks